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ONE-DIMENSIONAL CONSERVATION LAW WITH BOUNDARY CONDITIONS: GENERAL RESULTS AND SPATIALLY INHOMOGENEOUS CASE

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Abstract. The note presents the results of the recent work [5] of K. Sbihi and the author on existence and uniqueness of entropy solutions for boundary-value problem for conservation law $u_t + \varphi(u)_x = 0$ (here, we focus on the simplified one-dimensional setting). Then, using nonlinear semigroup theory, we extend these well-posedness results to the case of spatially dependent flux $\varphi(x, u)$.

1. **Introduction.** Consider the general boundary-value problem for one-dimensional conservation law in $Q := (0, T) \times \Omega$ where we choose $\Omega := (-\infty, 0)$:

$$\begin{cases} u_t + \varphi(x, u)_x = 0 & \text{in } \ Q_T := (0, T)\times(-\infty, 0) \\ u|_{t=0} = u_0 & \text{in } \ (-\infty, 0) \\ \varphi_\nu(u) \in \beta(u) & \text{on } \Sigma := (0, T)\times\{0\}. \end{cases} \qquad (E_{\varphi,\beta})$$

Here φ is a regular function of (x, u); $\varphi_\nu(\cdot)$ will denote $\varphi(0, \cdot)$; and β is a maximal monotone graph on $\mathbb{R}$ that encodes the boundary condition. The simplest and best known case is $\beta = \{u^D\} \times \mathbb{R}$, which encodes the Dirichlet condition $u = u^D$ on Σ.

The well-posedness theory of the Cauchy problem associated with the conservation law $u_t + \varphi(x, u)_x = f$ was achieved in the founding work of Kruzhkov [13]. Taking into account the boundary condition is a delicate matter. Indeed, already in the Dirichlet case, the classical work of Bardos, LeRoux and Nédélec [7] states that, for u_0 of bounded variation, there exists a unique entropy solution in Q to the conservation law in $(E_{\varphi,\beta})$ which satisfies a relaxed formulation of the boundary condition; this relaxed formulation is justified, as in [13], by the vanishing viscosity argument. We aim at explaining in which way the general boundary condition $\varphi_\nu(u) \in \beta(u)$ should be relaxed; this is of interest, e.g., for obstacle problems ($\beta = \partial I_{[m,M]}$ where ∂ is the subdifferential and I is the indicator function) and for the zero-flux boundary condition ($\beta = \{0\} \times \mathbb{R}$); the latter condition is particularly important in practice. Notice that our setting provides a nontrivial extension of the result of Bürger, Frid and Karlsen [9] on the zero-flux problem: we do not assume $\varphi(0) = 0 = \varphi(1)$. Thus the first objective of this note is to point out, in a simplified setting, the meaning that can be given to the *formal boundary condition* "$\varphi_\nu(u) \in \beta(u)$ on Σ". We highlight the ideas and results of the recent work [5] of K. Sbihi and the author, where the multi-dimensional problem with spatially homogeneous flux ($\varphi = \varphi(u)$) but variable graphs $\big(\beta_{(t,x)}\big)_{(t,x)\in\Sigma}$ was explored.

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The second objective of the note is to generalize some of the results of [5]. Indeed, the assumption of x-independence of the flux played an important role in the formulation, because it allowed to consider *strong traces* for u on Σ (see [3, 5] for details). In the present note, traces need not exist; the construction we use, proposed in [1], is based upon the nonlinear semigroup techniques (see [6]). It strongly relies on the assumption $N = 1$ and on the t-independence of both φ and β. The semigroup approach allows to bypass as well the usual technical assumption

$$\text{for a.e. } x \in \Omega,\ \varphi(x, \cdot) \text{ is non-affine on any interval } [a, b] \text{ with } a < b. \tag{H1}$$

What we prove is that there exists an entropy solution in $[0, T] \times \Omega$ which verifies well-chosen up-to-the boundary entropy inequalities (see Definition 2.2) involving β. We interpret the information contained in these up-to-the-boundary inequalities as the *effective boundary condition* "$\varphi_\nu(u) \in \widetilde{\mathcal{B}}(u)$". Here, the maximal monotone graph $\widetilde{\mathcal{B}}$ is the projection of β on the graph of the function φ_ν, as shown on Fig. 1.

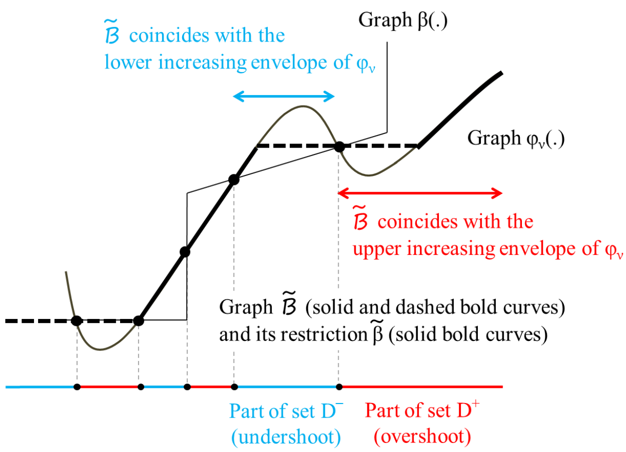

FIGURE 1. Construction of the projected graph $\widetilde{\mathcal{B}}$ and of $\widetilde{\beta}$

The main conclusion is: the graph β in the formulation of $(E_{\varphi,\beta})$ should be interpreted as its projection $\widetilde{\mathcal{B}}$. Indeed, the solution in the sense of Definition 2.2 can be attained as the limit of well-established approximation procedures (approximation of β by a kind of Yosida approximation or by "truncations" $\beta^{m,n} := \beta + I_{[-m,n]}$; the vanishing viscosity approximation involving the graph β or its approximates; and the Euler time-implicit discretization). Following Bardos, LeRoux and Nédélec [7], we see these facts as a justification of the notion of solution proposed for $(E_{\varphi,\beta})$.

2. Assumptions, definitions, results.

Definition 2.1. Extend β to a maximal monotone graph from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$ and define the *overshoot set* D^+ and the *undershoot set* D^- by

$$D^+ := \Big\{ z \in \overline{\mathbb{R}} \,|\, \sup \beta(z) \geq \varphi_\nu(z) \Big\}, \quad D^- := \Big\{ z \in \overline{\mathbb{R}} \,|\, \inf \beta(z) \leq \varphi_\nu(z) \Big\}.$$

Further, define the *crossing set* $D^0 := D^+ \cap D^-$. Finally, define $\widetilde{\mathcal{B}}$ on $\mathbb{R}$ as the closest to β maximal monotone graph that contains $\{(z, \varphi_\nu(z)) \,|\, z \in D^0\}$; and define $\widetilde{\beta}$ as the subgraph that $\widetilde{\mathcal{B}}$ and the graph $G\varphi_\nu$ of the function φ_ν have in common.

The "closest" to β graph $\widetilde{\mathcal{B}}$ does exist (see [5]). In fact, $\widetilde{\mathcal{B}}$ is single-valued and continuous, constituted of upper (respectively, lower) increasing envelopes of $G\varphi_\nu$ over connected components of D^+ (resp., D^-), see Fig. 1. It contains portions of $G\varphi_\nu$ complemented by horizontal segments over intervals of $\mathbb{R}\setminus \mathrm{Dom}\widetilde{\beta}$. As to $\widetilde{\beta}$, it is a maximal monotone subgraph of $G\varphi_\nu$. In the Dirichlet case ([7]), the graph $\tilde{\beta}$ appeared in the work [12] as a way to express the Bardos-LeRoux-Nédélec condition.

Now, write $q^\pm(x,u,k)$ for $\mathrm{sign}\,(u-k)(\varphi(x,u)-\varphi(x,k))$ ("semi-Kruzhkov" entropy fluxes). Following [13] and adapting the boundary approach of Carrillo [10], we set

Definition 2.2. An $L^\infty(Q)$ function u is called entropy solution of problem $(E_{\varphi,\beta})$ if $u(0,\cdot)=u_0$[1] and u verifies the following inequalities[2]:

$$\begin{gathered}\forall k\in\mathbb{R}\ \ \forall \xi\in\mathcal{D}((0,T)\times\overline{\Omega}),\ \xi\geq 0,\ \ \text{such that } \xi|_\Sigma=0 \text{ if } k\in D^\mp\\ \int_0^T\!\!\int_\Omega\Big(-(u-k)^\pm\xi_t-q^\pm(x,u,k)\cdot\nabla\xi\Big)+\int_0^T\!\!\int_\Omega \mathrm{sign}^\pm(u-k)\varphi_x(x,k)\,\xi\ \leq\ 0\end{gathered}\tag{1}$$

Let us clarify the relation between the *formal boundary condition* "$\varphi_\nu(u)\in\beta(u)$" and the condition contained in Definition 2.2. We claim that, up to technical details

$$\varphi_\nu(u)\in\widetilde{\mathcal{B}}(u)\ \text{ on }\Sigma\tag{2}$$

is the boundary relation entailed by inequalities (1).

Proposition 2.3. *(see [5, Prop. 3.3]) In the case where u admits a strong boundary trace[3] γu on Σ, u is an entropy solution in the sense of Definition 2.2 if and only if it verifies the Kruzhkov inequalities with $\xi\in\mathcal{D}((0,T)\times\Omega)$, $\xi\geq 0$, and*

$$(\gamma u)(t)\in Dom\,\widetilde{\beta}\ \text{ for a.e. } t\in(0,T).\tag{3}$$

Furthermore, in this situation (3) is equivalent to the property

$$\forall k\in D^\pm\quad q^\pm(0,(\gamma u)(t),k)\geq 0\ \text{ for a.e. } t\in(0,T).\tag{4}$$

Since $\widetilde{\beta}$ is a subgraph of the graph of φ_ν, relation (3) means that $\varphi_\nu(u)\in\widetilde{\beta}(u)$ (so that $\varphi_\nu(u)\in\widetilde{\mathcal{B}}(u)$); therefore we say that (3) (or (2)) is the *effective boundary condition* for problem $(E_{\varphi,\beta})$. Condition (3) was introduced by K. Sbihi in her thesis [15] (see also [3, 4]). For details on the graph $\widetilde{\mathcal{B}}$, on the entropy formulation (1) and its different reformulations, a detailed study of existence and convergence of approximate procedures we refer to the recent paper [5] of Sbihi and the author.

The results of [5] for the x-independent flux $\varphi\in C(\mathbb{R})$ in space dimension one can be summarized as follows. Consider the assumption

$$\exists A>0\ \ \forall z\notin[-A,A]\ \ \mathrm{sign}\,(z)\,\phi_\nu(z)\ \leq\ \mathrm{sign}\,z\,\beta(z).\tag{H2}$$

While (H2) is not required for well-posedness, we use it to get L^∞ estimate needed to prove that the vanishing viscosity method converges to the entropy solution in the sense (1). When (H2) is dropped, in order to justify the advent of $\tilde{\beta}$ we need an additional stage of approximation of β by rapidly growing at infinity graphs $\beta^{m,n}$.

Theorem 2.4. *(compilation of results of [5], one-dimensional case, $\varphi(x,u)\equiv\varphi(u)$)*
(i) (uniqueness, comparison, contraction) If $u,\hat{u}$ are solutions of $(E_{\varphi,\beta})$ in the sense of Definition 2.2 with initial data $u_0,\hat{u}_0$ respectively, then for a.e. $t\in(0,T)$

$$\|(u-\hat{u})^+(\cdot,t)\|_{L^1}\leq\|(u_0-\hat{u}_0)^+\|_{L^1}.\tag{5}$$

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[1] We have $u\in C(0,T;L^1_{loc}(\Omega))$ since by (1), u is a Kruzhkov solution inside Q (see, e.g., [5]).

[2] Note that admissible test functions ξ in (1) are different for the "+" sign and for the "−" sign.

[3] In different contexts, such solutions were called *trace-regular* in [1, 2].

(ii) (existence, construction of solution) Assume (H1). *Then for all L^∞ datum u_0 there exists a (unique) solution u of* ($E_{\varphi,\beta}$) *in the sense of Definition 2.2.*

If (H2) *holds, then $u = \lim_{\varepsilon\to 0} u^\varepsilon$; here u^ε is a weak solution to problem* ($E_{\varphi,\beta}$) *regularized by the vanishing viscosity term εu_{xx} (with "$\varphi_\nu(u) - \varepsilon u_x \in \beta(u)$" as boundary condition). And, if* (H2) *does not hold, then $u = \lim_{n,m\to\infty} u^{m,n}$ where $u^{m,n}$ is the vanishing viscosity limit for the boundary graph $\beta^{m,n} = \beta + I_{[-m,n]}$.*

A crucial point of the uniqueness proof is that strong boundary traces $\gamma u, \gamma\hat{u}$ on Σ exist (see [3]). Generalization of the uniqueness result to the multi-dimensional case and space-time dependent graphs β is straightforward, but the boundary condition cannot always be formulated using (1) (e.g., (3) can be used instead). Existence results for this general case involve technical assumptions which main goal is to ensure uniform L^∞ estimates on the approximate solutions. The assumption on nonlinearity of φ can be relaxed if φ is Lipschitz continuous (see [5, Th. 7.1]). Truncations $\beta^{m,n}$ can be replaced by a two-parameter Yosida approximation (see [5, Ex. 6.14]).

Now, let us look at the case where $\varphi = \varphi(x,u)$. Because we focus on understanding the boundary condition, we consider a single boundary point $x = 0$ and we avoid a few technical difficulties by taking the following (rather artificial) assumption:

$$\varphi(x,\cdot) \equiv 0 \ \text{ for } x \le -1. \tag{H3}$$

We also assume

$$\text{both } \varphi \text{ and } \varphi_x \text{ are Lipschitz continuous on } [-1,0]\times\mathbb{R}, \tag{H4}$$

$$\Phi : z \mapsto \max_{x\in[-1,1]} \varphi_x(x,z) \ \text{ is Lipschitz continuous on } [-1,0]\times\mathbb{R}. \tag{H5}$$

Assumptions (H4),(H5) can be relaxed; in particular, the role of (H5) is to ensure an L^∞ estimate on solutions in the situation where (H2) holds.

For x-dependent flux φ, existence of strong boundary traces for Kruzhkov entropy solutions of $u_t + \varphi(x,u)_x = 0$ in Q is yet not proved. Despite this obstacle, we will prove the following, which is the main new result of this note.

Theorem 2.5. *Assume* (H3),(H4),(H5) *hold. Assume $u_0 \in L^1(-\infty,0)\cap L^\infty(-\infty,0)$. Then there exists a unique entropy solution of* ($E_{\varphi,\beta}$) *in the sense of Definition 2.2.*

As in Theorem 2.4, the existence proof justifies the notion of solution (see Remark 1). The key ingredient for the proof of Theorem 2.5 is the *stationary problem*

$$\begin{cases} \hat{u} + \varphi(x,\hat{u})_x = f & \text{in } (-\infty,0) \\ \varphi_\nu(\hat{u}(0)) \in \beta(\hat{u}(0)). \end{cases} \tag{$S_{\varphi,\beta}$}$$

Problem ($S_{\varphi,\beta}$) is used as a building brick in construction of a solution of ($E_{\varphi,\beta}$) via the time-implicit discretization, and it is essential for the uniqueness proof. To state a notion of solution, consider that $\hat{u}$ is an entropy solution of ($S_{\varphi,\beta}$) if it is a time-independent solution of ($E_{\varphi,\beta}$) with additional source term $f = g - \hat{u}$.

3. Uniqueness, L^1 contraction and comparison proof: the ideas.

Using the Kruzhkov doubling of variables inside $[0,T]\times\Omega$, one gets[4] the *Kato inequality*

$$\int_\Omega \xi(u-\hat{u})^+(\cdot,t) \le \int_\Omega \xi(u_0-\hat{u}_0)^+ - \int_0^t\!\!\int_\Omega \nabla\xi\cdot q^+(x,u,\hat{u}) \tag{6}$$

with $\xi \in \mathcal{D}'(\Omega)$, $\xi \ge 0$, and for a.e. t. Here, we wish to let $\xi \to 1$ on Ω. If strong traces $\gamma u, \gamma\hat{u}$ on Σ exist, the last term passes to the limit and it yields the integral of

[4] If φ is x-dependent, this result is not entirely contained in [13]: see [1, Th. 5.1] for the full argument that relies on the fact that a *local* entropy solution is a vanishing viscosity limit.

$\operatorname{sign}^+(\gamma u - \gamma\hat{u})\big(\varphi_\nu(\gamma u) - \varphi_\nu(\gamma\hat{u})\big)$ over a part of Σ. Then, due to the characterization (3) and the monotonicity of $\widetilde{\beta}$, this term can be dropped and inequality (5) follows.

Now, in the situation where $\gamma\hat{u}$ exists but γu may not exist, we are still able to make the above arguments work. We use the following hint. Provided the strong trace $\gamma\hat{u}$ exists, the *weak trace* $\gamma_w q^\pm(\cdot, u(\cdot), \hat{u}(\cdot))$ on Σ (see [11]) verifies

$$\big(\gamma_w q^\pm(\cdot, u(\cdot), \hat{u}(\cdot))\big)(t) = \big(\gamma_w q^\pm(\cdot, u(\cdot), k)\big)(t)|_{k=(\gamma\hat{u})(t)}, \quad \text{for a.e. } t \in (0,T). \tag{7}$$

Furthermore, for a.e. t, $k = (\gamma\hat{u})(t) \in \operatorname{Dom}\widetilde{\beta}$ by Proposition 2.3. We also have

Lemma 3.1. *Assume u is an entropy solution of $(E_{\varphi,\beta})$ in the sense of Definition 2.2. Then for all $k \in D^\pm$, we have (respectively)*

$$(\gamma_w q^\pm(\cdot, u(\cdot), k))(t) \ge 0 \ \text{ for a.e. } t \in (0,T), \tag{8}$$

where γ_w denotes the weak boundary trace in the sense of Chen and Frid [11]. Furthermore, the two inequalities in (8) hold simultaneously for all $k \in \operatorname{Dom}\widetilde{\beta}$.

Proof. The first claim is straightforward. For the second one consider, e.g., $k \in D^+ \cap \operatorname{Dom}\widetilde{\beta}$. Then it is enough to prove $\gamma_w q^-(\cdot, u)\cdot), k) \ge 0$. To this end, take $k_0 \in [-\infty, k]$ such that k_0 is the closest to k point in D^-. By definition of $D^\pm$, we have $\varphi_\nu(\kappa) \le \varphi_\nu(k)$ for all $\kappa \in [k_0, k]$; hence $\varphi(x,\kappa) \le \varphi_\nu(k) + \overline{\overline{o}}_{x\to 0}(1)$. Developing the formula for $q(x, u(x), k)$, writing $\operatorname{sign}^-(u(x) - k) = 1\!\!1_{[u(x)\le k_0]} + 1\!\!1_{[k_0 < u(x) < k]}$, we find $q^-(x, u(x), k) \ge q^-(x, u(x), k_0) + \overline{\overline{o}}_{x\to 0}(1)$. Hence we can apply (8) with $k_0 \in D^-$ and deduce that $\gamma_w q^-(\cdot, u(\cdot), k) \ge 0$. Details can be found in [5, Prop. 7.4(i)]. ☐

Finally, combining (7) and Lemma 3.1, passing to the limit as $\xi \to 1$ in Kato inequality (6) we get the desired result (5) whenever $\hat{u}$ is *trace-regular*, i.e., $\gamma\hat{u}$ exists.

In order to put ourselves in the situation where $\hat{u}$ is trace-regular, we will consider $\hat{u} = \hat{u}(x)$ entropy solution of $(S_{\varphi,\beta})$. Actually, in the preceding argument we only need that the trace of the function $V\varphi_\nu(\hat{u})$ exist, where $V\varphi_\nu : z \mapsto \int_0^z |\varphi_\nu'(s)|\,ds$ is the variation function of φ_ν (also known as the *singular mapping*). We refer to [3, 5] for the use of $V\varphi_\nu$ within the arguments involving the traces of $q^\pm(\cdot, u(\cdot), \hat{u}(\cdot))$. Existence of traces for solutions of $(S_{\varphi,\beta})$ follows, roughly speaking, from the fact that $q(\cdot, \hat{u}(\cdot), k) \in W^{1,1}(-\infty, 0)$ for all $k \in \mathbb{R}$. We refer to [1, Lemma 3.1] for the proof in the case where φ_ν has finitely many extrema; the general case is similar.

Fortunately, comparison results concerning two solutions u and $\hat{u}$ of $(E_{\varphi,\beta})$ can be deduced from those concerning one solution $u \in L^1(Q) \cap L^\infty(Q)$ and all possible stationary solutions $v \in L^1(\Omega) \cap L^\infty(\Omega)$: to do this, one exploits the theory of nonlinear semigroups governed by m-accretive operators (details of this approach can be found in [8, 1, 2]). Roughly speaking, the above arguments prove that an entropy solution u of $(E_{\varphi,\beta})$ is an *integral solution* of the abstract evolution problem

$$\frac{d}{dt}u + \mathcal{A}u \ni 0, \quad u(0) = u_0 \tag{9}$$

where $\mathcal{A}$ is the operator associated with the formal expression $u(\cdot) \mapsto \varphi(\cdot, u(\cdot))_x$ in the entropy sense, as defined in (11) below. Then we apply the general result of uniqueness of an integral solution (see [6, 8] and Theorem 4.2 below).

4. Study of the stationary problem and use of the semigroup theory.

In the space $L^1 = L^1((-\infty, 0))$, consider the following definitions.

Definition 4.1 (elements of the nonlinear semigroup theory, see [6]).
• The *bracket* on L^1 is given by $\big[\, v\,,\, w\,\big] = \int w \operatorname{sign} v + \int w\, 1\!\!1_{[x\,|\,v(x)=0]}$.
• A multi-valued nonlinear operator $\mathcal{A}$ on L^1 is accretive if for all $(v,w),(\hat v,\hat w)\in\mathcal{A}$ one has $\big[v-\hat v, w-\hat w\big]_{L^1}\geq 0$. It is called m-accretive if, in addition, the domain $\mathrm{Dom}(I+\lambda\mathcal{A})^{-1}$ of the resolvent of $\mathcal{A}$ equals L^1 for all sufficiently small $\lambda>0$.
• A function $u\in C([0,T];L^1)$ is an integral solution of problem (9) if

$$\forall (v,w)\in\mathcal{A}\quad \frac{d}{dt}\|u(t)-v\|_{L^1}\leq \big[u(t)-v, 0-w\big]_{L^1}\ \text{ in } \mathcal{D}'((0,T)). \tag{10}$$

The main result associated with these notions is the following (see, e.g., [6]).

Theorem 4.2. *Assume that $\mathcal{A}$ is accretive and its closure, m-accretive; assume $Dom\,\mathcal{A}=L^1$. Then for all $u_0\in L^1$ there exists a unique integral solution to (9); further, two solutions with different data verify (5). Moreover, the integral solution is obtained by time-explicit discretization method (the Crandall-Liggett formula).*

Now, we apply the result to the operator $\mathcal{A}$ defined by its graph:

$$\begin{aligned}\mathcal{A}:=\{(\hat u,g)\in L^1\times L^1\,|\,&\hat u \text{ is an entropy solution of } (S_{\varphi,\beta}) \text{ with } f=\hat u+g,\\ &\text{in particular, } V\varphi_\nu(\hat u(\cdot)) \text{ is continuous at } x=0^-\}.\end{aligned} \tag{11}$$

Proposition 4.3 (properties of the stationary problem $(S_{\varphi,\beta})$).
(i) The operator $\mathcal{A}$ is accretive on L^1, moreover, its closure is m-accretive on L^1.
(ii) The domain $Dom\,\mathcal{A}$ is dense in L^1.

Proof. The accretivity in (i) follows by rewriting the arguments of the beginning of this section for stationary solutions with strong traces (to be precise, with those of $V\varphi_\nu(u)$). In the place of (5) we find the refined contraction property

$$\|u-\hat u\|_{L^1}\leq \int_\Omega \operatorname{sign}(u-\hat u)(f-\hat f)+\int_\Omega 1\!\!1_{[x\,|\,u(x)=\hat u(x)]}|f-\hat f| \tag{12}$$

which, together with the definition of $\mathcal{A}$, implies its accretivity in L^1.

Further, the m-accretivity of the closure of $\mathcal{A}$ is an existence claim for $(S_{\varphi,\beta})$ (with flux $\lambda\varphi$ and λ small enough) for some L^1-dense set of specific data. E.g., it is enough to prove that the set C^1_c of compactly supported in $(-\infty,0]$ functions of class C^1 is included in the domain $\mathrm{Dom}(I+\lambda\mathcal{A})^{-1}$ of the resolvent of $\mathcal{A}$, for all $\lambda>0$ small enough. This claim is proved using vanishing viscosity approximation.

First, we solve $u^\varepsilon+\lambda\varphi(x,u^\varepsilon)_x=\varepsilon u^\varepsilon_{xx}+f$ subject to the boundary condition $\varphi_\nu(u^\varepsilon)+\varepsilon u^\varepsilon_x\in\beta(u^\varepsilon)$ at $x=0$. Existence of a weak (variational) solution u^ε follows by adapting classical arguments (β can be regularized, then the problem is reduced to a coercive nonlinear elliptic problem in $H^1(-\infty,0)$ for which a solution can be constructed by a Leray-Schauder argument). Following Carrillo [10], we can get the comparison result analogous to (5), with $\varepsilon>0$.

Then compactness of the sequence $(u^\varepsilon)_\varepsilon$ should be obtained, and for this we need firstly a uniform L^∞ estimate on the solution. This estimate follows from the comparison of u^ε with constant functions. To see this, observe that $k\in\mathbb{R}^+$ is a solution of $k+\lambda\varphi(x,k)_x=k+\lambda\varphi_x(x,k)\geq 0=\varepsilon k_{xx}$ for λ small enough (here, assumption (H5) is used). Thus the constant k is a super-solution of the equation inside $(-\infty,0)$; the delicate point is to ensure that k is a super-solution of our viscosity regularized boundary-value problem with graph β. This is true for $k>A$ provided assumption (H2) holds; thus we temporarily assume (H2). In a similar way, we prove that $k < -A$ is a sub-solution, and by the comparison argument we find a uniform in ε bound in L^∞ on u^ε in terms of A and the right-hand side f.

Following [7], let us get a uniform BV estimate on $(u^\varepsilon)_\varepsilon$. For $v^\varepsilon := u^\varepsilon_x$ we have

$$(1+\lambda\varphi_{xu}(x,u^\varepsilon))v^\varepsilon + (\lambda\varphi_u(x,u^\varepsilon)v^\varepsilon - \varepsilon v^\varepsilon_x)_x = f_x - \lambda\varphi_{xx}(x,u^\varepsilon). \tag{13}$$

By (H3) and because f is compactly supported, $u^\varepsilon - \varepsilon u^\varepsilon_{xx} = 0$ for $x < -1$; since $\|u^\varepsilon\|_\infty \le const$, we get $|u^\varepsilon(x)| \le const\, e^{-|x|}$ for all x. Now, the flux of (13) is $F^\varepsilon := \lambda\varphi_u(\cdot,u^\varepsilon)v^\varepsilon - \varepsilon v^\varepsilon_x$; it verifies $F^\varepsilon(0) = \lambda\varphi_x(0,u^\varepsilon(0)) - \int_{-\infty}^0 (f-u^\varepsilon)(y)\,dy$. By (H4), $|F^\varepsilon(0)| \le const$ is bounded. Further, $1+\lambda\varphi_{xu}(\cdot,u^\varepsilon(\cdot)) \ge \frac{1}{2}$ for small enough λ, due to (H4). Now we take a Lipschitz approximation of $\operatorname{sign} v^\varepsilon$ as a test function; a uniform estimate of $\int_{-\infty}^0 |v^\varepsilon| = \|u^\varepsilon_x\|_{L^1}$ follows. Due to the exponential decay of u^ε at $-\infty$, we see that $(u^\varepsilon)_\varepsilon$ admits an accumulation point $u \in L^1$.

Now, it remains to write entropy inequalities for u^ε and pass to the limit. From the weak formulation, using Lipschitz approximations of $\operatorname{sign}^\pm(u^\varepsilon - k)$ as test functions (see, e.g., [10] and [1, Appendix]), for all $k \in \mathbb{R}$ we get

$$\begin{aligned}\int_\Omega \Big((u^\varepsilon-k)^\pm\xi - q^\pm(x,u^\varepsilon,k)\cdot\nabla\xi\Big) + \int_\Omega \operatorname{sign}^\pm(u^\varepsilon-k)\varphi_x(x,k)\xi \\ \le \int_\Omega \varepsilon|u^\varepsilon-k|_x\xi_x \;-\; \operatorname{sign}^\pm(u^\varepsilon(0)-k)(b^\varepsilon-\varphi_\nu(k))\xi(0)\end{aligned} \tag{14}$$

where $b^\varepsilon \in \beta(u^\varepsilon(0))$ (the last term is a boundary term). Convergence of u^ε and the classical uniform estimate of $\|\varepsilon(u^\varepsilon_x)^2\|_{L^1}$ permit to pass to the limit in all terms except for the last one, which we will bound from above. Consider, e.g., $k \in D^+$. In the "sign^+" inequality (14), the monotonicity of β and the choice of k yield

$$-\operatorname{sign}^+(u^\varepsilon(0)-k)(b^\varepsilon-\varphi_\nu(k)) \le (b(k)-\varphi_\nu(k))^- = 0. \tag{15}$$

Further, we simply impose $\xi(0)=0$ in the "sign^-" inequality (14) and the boundary term vanishes. Thus we arrive to the stationary analogue of inequalities (1) with the adequate choice of ξ. This proves that u is an entropy solution of $(S_{\varphi,\beta})$.

It remains to bypass (H2). This is done by working firstly with truncated graphs $\beta^{m,n}$ that do satisfy (H2). Convergence of the associated solutions $u^{m,n}$ to a limit u is ensured by monotonicity (see [4, 5]). Then, as in [4], one observes that the boundary conditions for graphs $\widetilde{\beta}^{m,n}$ pass to the limit (e.g., if φ_ν is monotone near $\pm\infty$, then $\widetilde{\mathcal{B}}^{m,n}$ coincide with $\widetilde{\mathcal{B}}$ for large enough n,m). To be specific, if $k \in D^{\pm;n,m}$ (the overshoot or undershoot set defined for graph $\beta^{m,n}$) then $k \in D^\pm$ for n,m large enough. Thus entropy inequalities for $u^{m,n}$ yield analogous inequalities for u.

As to the claim (ii), it can be proved by showing that as $\lambda \to 0$, the solution of $u + \lambda\mathcal{A} = f$ converges to f in L^1; see [1] for details corresponding to our case. □

With Theorem 4.2 and Prop. 4.3, we get the uniqueness claim of Theorem 2.5:

Proposition 4.4. *An entropy solution of* $(E_{\varphi,\beta})$ *is an integral solution of* (9) *with* $\mathcal{A}$ *defined by* (11). *In particular, there exists at most one entropy solution for given datum, and we have* (5) *for entropy solutions* $u, \hat{u}$ *with data* $u_0, \hat{u}_0$.

5. Existence of solution, justification of the effective boundary condition.

In order to prove existence of an entropy solution, we can restrict our attention to $L^1 \cap L^\infty$ data due to assumptions (H3),(H4). Let us give two existence arguments.

Under the genuine nonlinearity assumption (H1) on $\varphi(x,\cdot)$, one can follow closely the existence proof of Proposition 4.3(i), substituting the stationary problem by the evolution problem. Indeed, (H5) (along with (H2)) allows to construct super- and sub-solutions of ($E_{\varphi,\beta}$) under the form $\hat{u}(t,x)=k(t)$. The uniform L^∞ bound along with assumption (H1) ensures compactness of $(u^\varepsilon)_\varepsilon$ in L^1_{loc} (see Panov [14]). In this argument, we do not need Lipschitz regularity of φ_x in (H4).

In general we do not assume (H1); we exploit the existence result for ($S_{\varphi,\beta}$) and the Crandall-Liggett construction of Theorem 4.2. Indeed, in this case the time compactness comes for gratis; one only has to show that the integral solution (also known as the mild solution) coming from Crandall-Liggett formula is also an entropy solution (cf. [6]). This itinerary was taken in the work of K. Sbihi ([15], see also [3]). In the setting of the present note, the proof is much simpler than in [15, 3] since the stability of the entropy formulation (1) by L^1 convergence is evident.

Thus we achieve the following result and complete the proof of Theorem 2.5:

Proposition 5.1. *For all $u_0 \in L^1((-\infty,0))\cap L^\infty((-\infty,0))$ there exists the integral solution to (9) which is also the unique entropy solution of ($E_{\varphi,\beta}$). For general L^∞ datum u_0, there exists a unique entropy solution of ($E_{\varphi,\beta}$) obtained as the $L^1_{loc}(Q)$ limit of solutions u_n with $L^1\cap L^\infty$ data $u_{0,n}(\cdot):=u_0(\cdot)1\!\!1_{[-n,0]}(\cdot)$.*

Remark 1. To conclude the note, let us stress that the entropy formulation (1) and the projected graph $\widetilde{\mathcal{B}}$ naturally appeared from the vanishing viscosity approximation of ($E_{\varphi,\beta}$) or ($S_{\varphi,\beta}$): the main arguments here were (14), (15), and Prop. 2.3.

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